POSITIVE SOLUTIONS TO A SUPERCRITICAL ELLIPTIC PROBLEM WHICH CONCENTRATE ALONG A THIN SPHERICAL HOLE

MÓNICA CLAPP, JORGE FAYA, AND ANGELA PISTOIA

ABSTRACT. We consider the supercritical problem

$$-\Delta v = |v|^{p-2}\, v \quad \text{in } \Theta_\epsilon, \qquad v = 0 \quad \text{on } \partial\Theta_\epsilon,$$

where Θ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $p > 2^* := \frac{2N}{N-2}$, and Θ_ϵ is obtained by deleting the ϵ-neighborhood of some sphere which is embedded in Θ. In some particular situations we show that, for $\epsilon > 0$ small enough, this problem has a positive solution v_ϵ and that these solutions concentrate and blow up along the sphere as $\epsilon \to 0$.

Our approach is to reduce this problem to a critical problem of the form

$$-\Delta u = Q(x)\, |u|^{\frac{4}{n-2}}\, u \quad \text{in } \Omega_\epsilon, \qquad u = 0 \quad \text{on } \partial\Omega_\epsilon,$$

in a punctured domain $\Omega_\epsilon := \{x \in \Omega : |x - \xi_0| > \epsilon\}$ of lower dimension, by means of some Hopf map. We show that, if Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, $\xi_0 \in \Omega$, $Q \in C^2(\overline{\Omega})$ is positive and $\nabla Q(\xi_0) \neq 0$ then, for $\epsilon > 0$ small enough, this problem has a positive solution u_ϵ, and that these solutions concentrate and blow up at ξ_0 as $\epsilon \to 0$.

KEY WORDS: Nonlinear elliptic problem; supercritical problem; nonautonomous critical problem; positive solutions; domains with a spherical perforation, blow-up along a sphere.

MSC2010: 35J60, 35J20.

1. Introduction

We are interested in the supercritical problem

$$-\Delta v = |v|^{p-2}\, v \quad \text{in } \mathcal{D}, \qquad v = 0 \quad \text{on } \partial\mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $p > 2^*$, with $2^* := \frac{2N}{N-2}$ the critical Sobolev exponent.

Existence of a solution to this problem is a delicate issue. Pohozhaev's identity [20] implies that (1.1) does not have a nontrivial solution if $\mathcal{D}$ is strictly starshaped and $p \geq 2^*$. On the other hand, Kazdan and Warner [10] showed that infinitely many radial solutions exist for every $p \in (2, \infty)$ if $\mathcal{D}$ is an annulus. For $p = 2^*$ Bahri and Coron [2] established the existence of at least one positive solution to problem (1.1) in every domain $\mathcal{D}$ having nontrivial reduced homology with $\mathbb{Z}/2$-coefficients. However, in the supercritical case this is not enough to guarantee existence. In fact, for each $1 \leq k \leq N-3$, Passaseo [18, 19] exhibited domains having the homotopy

Date: March 3, 2013.

M. Clapp and J. Faya are supported by CONACYT grant 129847 and PAPIIT grant IN106612 (Mexico). A. Pistoia is supported by Università degli Studi di Roma "La Sapienza" Accordi Bilaterali "Esistenza e proprietà geometriche di soluzioni di equazioni ellittiche non lineari" (Italy).

type of a k-dimensional sphere in which problem (1.1) does not have a nontrivial solution for $p \geq 2^*_{N,k} := \frac{2(N-k)}{N-k-2}$. Existence may fail even in domains with richer topology, as shown in [4].

The first nontrivial existence result for $p > 2^*$ was obtained by del Pino, Felmer and Musso [5] in the slightly supercritical case, i.e. for $p > 2^*$ but close enough to 2^*. For p slightly below $2^*_{N,1}$ solutions in certain domains, concentrating at a boundary geodesic as $p \to 2^*_{N,1}$, were constructed in [7].

A fruitful approach to produce solutions to the supercritical problem (1.1) is to reduce it to some critical or subcritical problem in a domain of lower dimension, either by considering rotational symmetries, or by means of maps which preserve the laplacian, or by a combination of both. This approach has been recently taken in [1, 4, 11, 12, 15, 21] to produce solutions of (1.1) in different types of domains. We shall also follow this approach to obtain a new type of solutions in domains with thin spherical perforations.

We start with some notation. Let $O(N)$ be the group of linear isometries of $\mathbb{R}^N$. If Γ is a closed subgroup of $O(N)$, we denote by $\Gamma x := \{gx : g \in \Gamma\}$ the Γ-orbit of $x \in \mathbb{R}^N$. A domain $\mathcal{D}$ in $\mathbb{R}^N$ is called Γ-invariant if $\Gamma x \subset \mathcal{D}$ for all $x \in \mathcal{D}$, and a function $u : \mathcal{D} \to \mathbb{R}$ is called Γ-invariant if u is constant on every Γx. We denote by

$$\mathcal{D}^\Gamma := \{x \in \mathcal{D} : gx = x \quad \forall g \in \Gamma\}$$

the set of Γ-fixed points in $\mathcal{D}$.

We consider the problem

$$(\wp^*_{Q,\epsilon}) \qquad \begin{cases} -\Delta u = Q(x) u^{\frac{n+2}{n-2}} & \text{in } \Omega_\epsilon, \\ u > 0 & \text{in } \Omega_\epsilon, \\ u = 0 & \text{on } \partial\Omega_\epsilon, \end{cases}$$

in

$$\Omega_\epsilon := \{x \in \Omega : |x - \xi_0| > \epsilon\},$$

where $n \geq 3$, Ω is a bounded smooth domain in $\mathbb{R}^n$ which is invariant under the action of some closed subgroup Γ of $O(n)$, $\xi_0 \in \Omega^\Gamma$, and the function $Q \in C^2(\overline{\Omega})$ is Γ-invariant and satisfies $\min_{x \in \overline{\Omega}} Q(x) > 0$. Note that, since $\xi_0 \in \Omega^\Gamma$, Ω_ϵ is also Γ-invariant.

We will prove the following result.

Theorem 1.1. *Assume that $\nabla Q(\xi_0) \neq 0$. Then there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem $(\wp^*_{Q,\epsilon})$ has a Γ-invariant solution u_ϵ which concentrates and blows up at the point ξ_0 as $\epsilon \to 0$.*

Now we describe two situations where one can apply this result to obtain solutions of supercritical problems which concentrate and blow up at a sphere.

For $N = 2, 4, 8, 16$ we write $\mathbb{R}^N = \mathbb{K} \times \mathbb{K}$, where $\mathbb{K}$ is either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$. The set of units $\mathbb{S}_\mathbb{K} := \{\vartheta \in \mathbb{K} : |\vartheta| = 1\}$, which is a group if $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and a quasigroup with unit if $\mathbb{K} = \mathbb{O}$, acts on $\mathbb{R}^N$ by multiplication on each coordinate, i.e. $\vartheta(z_1, z_2) := (\vartheta z_1, \vartheta z_2)$. The orbit space of $\mathbb{R}^N$ with respect to this action turns out to be $\mathbb{R}^{\dim \mathbb{K}+1}$ and the projection onto the orbit space is the Hopf map $\mathfrak{h}_\mathbb{K} : \mathbb{R}^N = \mathbb{K} \times \mathbb{K} \to \mathbb{R} \times \mathbb{K} = \mathbb{R}^{\dim \mathbb{K}+1}$ given by

$$\mathfrak{h}_\mathbb{K}(z_1, z_2) := (|z_1|^2 - |z_2|^2,\, 2\overline{z_1} z_2).$$

What makes this map special is that it preserves the laplacian. Maps with this property are called harmonic morphisms [3, 8, 23]. More precisely, the following statement holds true. It can be derived by straightforward computation (cf. Proposition 4.1) or from the general theory of harmonic morphisms as in [4].

Proposition 1.2. *Let $N = 2, 4, 8, 16$ and let $\mathcal{D}$ be an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{R}^N = \mathbb{K}^2$ such that $0 \notin \overline{\mathcal{D}}$. Set $\mathcal{U} := \mathfrak{h}_{\mathbb{K}}(\mathcal{D})$. If u is a solution to problem*

$$\begin{cases} -\Delta u = \frac{1}{2|x|}\,|u|^{p-2}\,u & \text{in } \mathcal{U}, \\ u = 0 & \text{on } \partial\mathcal{U}, \end{cases} \tag{1.2}$$

then $v := u \circ \mathfrak{h}_{\mathbb{K}}$ is an $\mathbb{S}_{\mathbb{K}}$-invariant solution of problem (1.1). *Conversely, if v is an $\mathbb{S}_{\mathbb{K}}$-invariant solution of problem* (1.1) *and $v = u \circ \mathfrak{h}_{\mathbb{K}}$, then u solves* (1.2).

We apply this result as follows: Let $N = 4, 8, 16$ and let Θ be an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{R}^N = \mathbb{K}^2$ such that $0 \notin \overline{\Theta}$. Fix a point $z_0 \in \Theta$ and for each $\epsilon > 0$ small enough let

$$\Theta_\epsilon := \{z \in \Theta : \operatorname{dist}(z, \mathbb{S}_{\mathbb{K}} z_0) > \epsilon\}$$

where $\mathbb{S}_{\mathbb{K}} z_0 := \{\vartheta z : \vartheta \in \mathbb{S}_{\mathbb{K}}\}$. This is again an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{K}^2$. We consider the supercritical problem

$$\begin{cases} -\Delta v = v^{\frac{\dim \mathbb{K}+3}{\dim \mathbb{K}-1}} & \text{in } \Theta_\epsilon, \\ v > 0 & \text{in } \Theta_\epsilon, \\ u = 0 & \text{on } \partial\Theta_\epsilon. \end{cases} \tag{$\wp^1_\epsilon$}$$

Then, Theorem 1.1 with $n := \dim \mathbb{K} + 1$, $\Gamma = \{1\}$, $\Omega := \mathfrak{h}_{\mathbb{K}}(\Theta)$, $\xi_0 := \mathfrak{h}_{\mathbb{K}}(z_0)$ and $Q(x) := \frac{1}{2|x|}$, together with Proposition 1.2, immediately yields the following result.

Theorem 1.3. *There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, the supercritical problem $(\wp^1_\epsilon)$ has an $\mathbb{S}_{\mathbb{K}}$-invariant solution v_ϵ which concentrates and blows up along the sphere $\mathbb{S}_{\mathbb{K}} z_0$ as $\epsilon \to 0$.*

Now let $O(m) \times O(m)$ act on $\mathbb{R}^{2m} \equiv \mathbb{R}^m \times \mathbb{R}^m$ in the obvious way and $O(m)$ act on the last m coordinates of $\mathbb{R}^{m+1} \equiv \mathbb{R} \times \mathbb{R}^m$. We write the elements of $\mathbb{R}^{2m}$ as (y_1, y_2) with $y_i \in \mathbb{R}^m$ and the elements of $\mathbb{R}^{m+1}$ as $x = (t, \zeta)$ with $t \in \mathbb{R}$, $\zeta \in \mathbb{R}^m$. Recently Pacella and Srikanth showed that the real Hopf map provides a one-to-one correspondence between $[O(m) \times O(m)]$-invariant solutions of a supercritical problem in a domain in $\mathbb{R}^{2m}$ and $O(m)$-invariant solutions of a critical problem in some domain in $\mathbb{R}^{m+1}$. In [16] they proved the following result.

Proposition 1.4. *Let $N = 2m$, $m \geq 2$, and $\mathcal{D}$ be an $[O(m) \times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$ such that $0 \notin \overline{\mathcal{D}}$. Set*

$$\mathcal{U} := \{(t, \zeta) \in \mathbb{R} \times \mathbb{R}^m : \mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|) = (t, |\zeta|) \textit{ for some } (y_1, y_2) \in \mathcal{D}\}.$$

If $u(t, \zeta) = \mathfrak{u}(t, |\zeta|)$ is an $O(m)$-invariant solution of problem

$$\begin{cases} -\Delta u = \frac{1}{2|x|}\,|u|^{p-2}\,u & \text{in } \mathcal{U}, \\ u = 0 & \text{on } \partial\mathcal{U}, \end{cases} \tag{1.3}$$

then $v(y_1, y_2) := \mathfrak{u}(\mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|))$ is an $[O(m) \times O(m)]$-invariant solution of problem (1.1).

Conversely, if $v(y_1, y_2) = \mathfrak{v}(|y_1|, |y_2|)$ is an $[O(m) \times O(m)]$-invariant solution of problem (1.1) *and $\mathfrak{v} = \mathfrak{u} \circ \mathfrak{h}_{\mathbb{R}}$, then $u(t, \zeta) = \mathfrak{u}(t, |\zeta|)$ is an $O(m)$-invariant solution of problem* (1.2).

We apply this result as follows: Let Φ be an $[O(m)\times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$ such that $0\notin\overline{\Phi}$ and $(y_0,0)\in\Phi$. We write $S_0^{m-1}:=\{(y,0):|y|=|y_0|\}$ for the $[O(m)\times O(m)]$-orbit of $(y_0,0)$, and for each $\epsilon>0$ small enough we set

$$\Phi_\epsilon:=\{x\in\Phi:\operatorname{dist}(x,S_0^{m-1})>\epsilon\}.$$

This is again an $[O(m)\times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$. We consider the supercritical problem

$$(\wp_\epsilon^2)\qquad\begin{cases}-\Delta v=v^{\frac{m+3}{m-1}} & \text{in } \Phi_\epsilon,\\ v>0 & \text{in } \Phi_\epsilon,\\ u=0 & \text{on } \partial\Phi_\epsilon.\end{cases}$$

Then, Theorem 1.1 with $n=m+1$, $\Gamma=O(m)$,

$$\Omega:=\{(t,\zeta)\in\mathbb{R}\times\mathbb{R}^m:\mathfrak{h}_{\mathbb{R}}(|y_1|,|y_2|)=(t,|\zeta|)\text{ for some }(y_1,y_2)\in\Phi\},$$

$\xi_0:=(|y_0|,0,\dots,0)$ and $Q(x)=\frac{1}{2|x|}$, together with Proposition 1.4, immediately yields the following result.

Theorem 1.5. *There exists $\epsilon_0>0$ such that, for each $\epsilon\in(0,\epsilon_0)$, problem $(\wp_\epsilon)$ has an $[O(m)\times O(m)]$-invariant solution v_ϵ which concentrates and blows up along the $(m-1)$-dimensional sphere S_0^{m-1} as $\epsilon\to 0$.*

The proof of Theorem 1.1 uses the well-known Ljapunov-Schmidt reduction, adapted to the symmetric case. In the following section we sketch this reduction, highlighting the places where the symmetries play a role. In section 3 we give an expansion of the reduced energy functional and use it to prove Theorem 1.1. We conclude with some remarks concerning Proposition 1.4.

2. The finite dimensional reduction

For every bounded domain $\mathcal{U}$ in $\mathbb{R}^n$ we take

$$(u,v):=\int_{\mathcal{U}}\nabla u\cdot\nabla v,\qquad \|u\|:=\left(\int_{\mathcal{U}}|\nabla u|^2\right)^{1/2},$$

as the inner product and its corresponding norm in $H_0^1(\mathcal{U})$. If we replace $\mathcal{U}$ by $\mathbb{R}^n$ these are the inner product and the norm in $D^{1,2}(\mathbb{R}^n)$. We write

$$\|u\|_r:=(\int_{\mathcal{U}}|u|^r)^{1/r}$$

for the norm in $L^r(\mathcal{U})$, $r\in[1,\infty)$.

If $\mathcal{U}$ is Γ-invariant for some closed subgroup Γ of $O(n)$ we set

$$H_0^1(\mathcal{U})^\Gamma:=\{u\in H_0^1(\mathcal{U}):u\text{ is }\Gamma\text{-invariant}\}$$

and, similarly, for $D^{1,2}(\mathbb{R}^n)^\Gamma$ and $L^r(\mathcal{U})^\Gamma$.

It is well known that the standard bubbles

$$U_{\delta,\xi}(x)=[n(n-2)]^{\frac{n-2}{4}}\frac{\delta^{\frac{n-2}{2}}}{(\delta^2+|x-\xi|^2)^{\frac{n-2}{2}}}\qquad \delta\in(0,\infty),\quad \xi\in\mathbb{R}^n,$$

are the only positive solutions of the equation

$$-\Delta U=U^p\quad\text{in }\ \mathbb{R}^n,$$

where $p := \frac{n+2}{n-2}$. Thus, the function $W_{\delta,\xi} := \gamma_0 U_{\delta,\xi}$, with $\gamma_0 := [Q(\xi_0)]^{\frac{-1}{p-1}}$, solves the equation

$$-\Delta W = Q(\xi_0) W^p \qquad \text{in } \mathbb{R}^n. \tag{2.1}$$

Let

$$\psi^0_{\delta,\xi} := \frac{\partial U_{\delta,\xi}}{\partial \delta} = \alpha_n \frac{n-2}{2} \delta^{\frac{n-4}{2}} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{n/2}}, \tag{2.2}$$

$$\psi^j_{\delta,\xi} := \frac{\partial U_{\delta,\xi}}{\partial \xi_j} = \alpha_n (n-2) \delta^{\frac{n-2}{2}} \frac{x_j - \xi_j}{(\delta^2 + |x-\xi|^2)^{n/2}}, \qquad j = 1, \dots, n.$$

The space generated by these $n+1$ functions is the space of solutions to the problem

$$-\Delta \psi = p U^{p-1}_{\delta,\xi} \psi, \qquad \psi \in D^{1,2}(\mathbb{R}^n). \tag{2.3}$$

Note that

$$U_{\delta,\xi} \in D^{1,2}(\mathbb{R}^n)^\Gamma \qquad \text{iff} \qquad \xi \in (\mathbb{R}^n)^\Gamma$$

and, similarly, for every $j = 0, 1, \dots, n$,

$$\psi^j_{\delta,\xi} \in D^{1,2}(\mathbb{R}^n)^\Gamma \qquad \text{iff} \qquad \xi \in (\mathbb{R}^n)^\Gamma.$$

Let Ω be a Γ-invariant bounded smooth domain in $\mathbb{R}^n$, $Q \in C^2(\overline{\Omega})$ be positive and Γ-invariant, and $\xi_0 \in \Omega^\Gamma$. For $\epsilon > 0$ small enough set

$$\Omega_\epsilon := \{x \in \Omega : |x - \xi_0| > \epsilon\}.$$

Consider the orthogonal projection $P_\epsilon : D^{1,2}(\mathbb{R}^n) \to H^1_0(\Omega_\epsilon)$, i.e. if $W \in D^{1,2}(\mathbb{R}^n)$ then $P_\epsilon W$ is the unique solution to the problem

$$-\Delta (P_\epsilon W) = -\Delta W \quad \text{in } \Omega_\epsilon, \qquad P_\epsilon W = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.4}$$

A consequence of the uniqueness is that $P_\epsilon W \in H^1_0(\Omega_\epsilon)^\Gamma$ if $W \in D^{1,2}(\mathbb{R}^n)^\Gamma$.

We denote by $G(x,y)$ the Green function of the Laplace operator in Ω with zero Dirichlet boundary condition and by $H(x,y)$ its regular part, i.e.

$$G(x,y) = \beta_n \left(\frac{1}{|x-y|^{n-2}} - H(x,y) \right),$$

where β_n is a positive constant depending only on n. The following estimates will play a crucial role in the proof of Theorem 1.1.

Lemma 2.1. *Assume that $\delta \to 0$ as $\epsilon \to 0$ and $\epsilon = o(\delta)$ as $\epsilon \to 0$. Fix $\eta \in \mathbb{R}^n$, set $\xi := \xi_0 + \delta\eta$, and define*

$$R(x) := P_\epsilon U_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \alpha_n \delta^{\frac{n-2}{2}} H(x,\xi) + \frac{\alpha_n}{\delta^{\frac{n-2}{2}} (1 + |\eta|^2)^{\frac{n-2}{2}}} \frac{\epsilon^{n-2}}{|x-\xi_0|^{n-2}}.$$

Then there exists a positive constant c such that the following estimates hold true for every $x \in \Omega \smallsetminus B(\xi_0, \epsilon)$:

$$|R(x)| \le c\delta^{\frac{n-2}{2}} \left[\frac{\epsilon^{n-2}(1 + \epsilon\delta^{-n+1})}{|x-\xi_0|^{n-2}} + \delta^2 + \left(\frac{\epsilon}{\delta}\right)^{n-2} \right],$$

$$|\partial_\delta R(x)| \le c\delta^{\frac{n-4}{2}} \left[\frac{\epsilon^{n-2}(1 + \epsilon\delta^{-n+1})}{|x-\xi_0|^{n-2}} + \delta^2 + \left(\frac{\epsilon}{\delta}\right)^{n-2} \right],$$

$$|\partial_{\xi_i} R(x)| \le c\delta^{\frac{n}{2}} \left[\frac{\epsilon^{n-2}(1 + \epsilon\delta^{-n})}{|x-\xi_0|^{n-2}} + \delta^2 + \frac{\epsilon^{n-2}}{\delta^{n-1}} \right].$$

Proof. See Lemma 3.1 in [9]. $\square$

For each $\epsilon > 0$ and $(d, \eta) \in \Lambda^\Gamma := (0, \infty) \times (\mathbb{R}^n)^\Gamma$ set (see (2.1))

$$V_{d,\eta} := P_\epsilon W_{\delta,\xi} = \gamma_0 P_\epsilon U_{\delta,\xi} \qquad \text{with } \ \delta := d\epsilon^{\frac{n-2}{n-1}}, \ \ \xi := \xi_0 + \delta\eta.$$

The map $(d, \eta) \mapsto V_{d,\eta}$ is a C^2-embedding of Λ^Γ as a submanifold of $H_0^1(\Omega_\epsilon)^\Gamma$, whose tangent space at $V_{d,\eta}$ is

$$K^\epsilon_{d,\eta} := \operatorname{span}\{P_\epsilon \psi^j_{\delta,\xi} : j = 0, 1, \dots, n\}.$$

Note that, since $\xi_0, \eta \in (\mathbb{R}^n)^\Gamma$, also $\xi \in (\mathbb{R}^n)^\Gamma$ and, therefore, $K^\epsilon_{d,\eta} \subset H_0^1(\Omega_\epsilon)^\Gamma$. We write

$$K^{\epsilon,\perp}_{d,\eta} := \{\phi \in H_0^1(\Omega_\epsilon)^\Gamma : (\phi, P_\epsilon \psi^j_{\delta,\xi}) = 0 \ \text{ for } j = 0, 1, \dots, n\}$$

for the orthogonal complement of $K^\epsilon_{d,\eta}$ in $H_0^1(\Omega_\epsilon)^\Gamma$, and $\Pi^\epsilon_{d,\eta} : H_0^1(\Omega_\epsilon)^\Gamma \to K^\epsilon_{d,\eta}$ and $\Pi^{\epsilon,\perp}_{d,\eta} : H_0^1(\Omega_\epsilon)^\Gamma \to K^{\epsilon,\perp}_{d,\eta}$ for the orthogonal projections, i.e.

$$\Pi^\epsilon_{d,\eta}(u) := \sum_{j=0}^n (u, P_\epsilon \psi^j_{\delta,\xi}) P_\epsilon \psi^j_{\delta,\xi}, \qquad \Pi^{\epsilon,\perp}_{d,\eta}(u) := u - \Pi^\epsilon_{d,\eta}(u).$$

Let $i^*_\epsilon : L^{\frac{2n}{n+2}}(\Omega_\epsilon) \to H_0^1(\Omega_\epsilon)$ be the adjoint operator to the embedding $i_\epsilon : H_0^1(\Omega_\epsilon) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega_\epsilon)$, i.e. $v = i^*_\epsilon(u)$ if and only if

$$(v, \varphi) = \int_{\Omega_\epsilon} u\varphi \qquad \forall \varphi \in C_c^\infty(\Omega_\epsilon)$$

if and only if

$$-\Delta v = u \quad \text{in } \Omega_\epsilon, \qquad v = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.5}$$

Sobolev's inequality yields a constant $c > 0$, independent of ϵ, such that

$$\|i^*_\epsilon(u)\| \le c \,\|u\|_{\frac{2n}{n+2}} \qquad \forall u \in L^{\frac{2n}{n+2}}(\Omega_\epsilon), \ \ \forall \epsilon > 0. \tag{2.6}$$

Note again that

$$i^*_\epsilon(u) \in H_0^1(\Omega_\epsilon)^\Gamma \qquad \text{if } \ u \in L^{\frac{2n}{n-2}}(\Omega_\epsilon)^\Gamma.$$

We rewrite problem $(\wp^*_{Q,\epsilon})$ in the following equivalent way:

$$\begin{cases} u = i^*_\epsilon\left[Q(x) f(u)\right], \\ u \in H_0^1(\Omega_\epsilon), \end{cases} \tag{2.7}$$

where $f(s) := (s^+)^p$ and $p := \frac{n+2}{n-2}$.

We shall look for a solution to problem (2.7) of the form

$$u_\epsilon = V_{d,\eta} + \phi \qquad \text{with } (d, \eta) \in \Lambda^\Gamma \text{ and } \phi \in K^{\epsilon,\perp}_{d,\eta}. \tag{2.8}$$

As usual, our goal will be to find $(d, \eta) \in \Lambda^\Gamma$ and $\phi \in K^{\epsilon,\perp}_{d,\eta}$ such that, for ϵ small enough,

$$\Pi^{\epsilon,\perp}_{d,\eta}[V_{d,\eta} + \phi - i^*_\epsilon(Qf(V_{d,\eta} + \phi))] = 0 \tag{2.9}$$

and

$$\Pi^\epsilon_{d,\eta}[V_{d,\eta} + \phi - i^*_\epsilon(Qf(V_{d,\eta} + \phi))] = 0. \tag{2.10}$$

First we will show that, for every $(d,\eta)\in\Lambda^\Gamma$ and ϵ small enough, there exists an unique $\phi\in K^{\epsilon,\perp}_{d,\eta}$ which satisfies (2.9). To this aim we consider the linear operator $L^\epsilon_{d,\eta}:K^{\epsilon,\perp}_{d,\eta}\to K^{\epsilon,\perp}_{d,\eta}$ defined by

$$L^\epsilon_{d,\eta}(\phi):=\phi-\Pi^{\epsilon,\perp}_{d,\eta}\, i^*_\epsilon[Qf'(V_{d,\eta})\phi].$$

It has the following properties.

Proposition 2.2. *For every compact subset D of Λ^Γ there exist $\epsilon_0>0$ and $c>0$ such that, for each $\epsilon\in(0,\epsilon_0)$ and each $(d,\eta)\in D$,*

$$\left\|L^\epsilon_{d,\eta}(\phi)\right\|\geq c\,\|\phi\| \qquad \text{for all } \phi\in K^{\epsilon,\perp}_{d,\eta}, \tag{2.11}$$

and the operator $L^\epsilon_{d,\eta}$ is invertible.

Proof. The argument given in [9] to prove Lemma 5.1 carries over with minor changes to our situation. □

The following estimates may be found in [13].

Lemma 2.3. *For each $a,b,q\in\mathbb{R}$ with $a\geq 0$ and $q\geq 1$ there exists a positive constant c such that the following inequalities hold*

$$\left||a+b|^q-a^q\right|\leq\begin{cases} c\min\{|b|^q, a^{q-1}|b|\} & \text{if } 0<q<1,\\ c(|a|^{q-1}|b|+|b|^q) & \text{if } q\geq 1.\end{cases}$$

Again, the argument given to prove similar results in the literature carries over with minor changes to prove the following result. We include it this time to illustrate this fact and also because some of the estimates will be used later on.

Proposition 2.4. *For every compact subset D of Λ^Γ there exist $\epsilon_0>0$ and $c>0$ such that, for each $\epsilon\in(0,\epsilon_0)$ and for each $(d,\eta)\in D$, there exists a unique $\phi^\epsilon_{d,\eta}\in K^{\epsilon,\perp}_{d,\eta}\subset H^1_0(\Omega_\epsilon)^\Gamma$ which solves equation (2.9) and satisfies*

$$\left\|\phi^\epsilon_{d,\eta}\right\|\leq c\epsilon^{\frac{n-2}{n-1}}. \tag{2.12}$$

Moreover, the function $(d,\eta)\mapsto\phi^\epsilon_{d,\eta}$ is a C^1-map.

Proof. Note that $\phi\in K^{\epsilon,\perp}_{d,\eta}$ solves equation (2.9) if and only if ϕ is a fixed point of the operator $T^\epsilon_{d,\eta}:K^{\epsilon,\perp}_{d,\eta}\to K^{\epsilon,\perp}_{d,\eta}$ defined by

$$T^\epsilon_{d,\eta}(\phi)=(L^\epsilon_{d,\eta})^{-1}\Pi^{\epsilon,\perp}_{d,\eta} i^*_\epsilon\left[Qf(V_{d,\eta}+\phi)-Qf'(V_{d,\eta})\phi-Q(\xi_0)(\gamma_0U_{\delta,\xi})^p\right].$$

We will prove that $T^\epsilon_{d,\eta}$ is a contraction on a suitable ball.

To this aim, we first show that there exist $\epsilon_0>0$ and $c>0$ such that for, each $\epsilon\in(0,\epsilon_0)$,

$$\|\phi\|\leq c\epsilon^{\frac{n-2}{n-1}} \quad\Rightarrow\quad \left\|T^\epsilon_{d,\eta}(\phi)\right\|\leq c\epsilon^{\frac{n-2}{n-1}}. \tag{2.13}$$

From Proposition 2.2 we have that, for some $c>0$ and ϵ small enough,

$$\left\|(L^\epsilon_{d,\eta})^{-1}\right\|\leq c \qquad \forall (d,\eta)\in D.$$

Using (2.6) we obtain

$$\begin{aligned}\left\|T^{\epsilon}_{d,\eta}(\phi)\right\| &\le c\left\|Q\left[f(V_{d,\eta}+\phi)-f'(V_{d,\eta})\phi\right]-Q(\xi_0)(\gamma_0U_{\delta,\xi})^p\right\|_{\frac{2n}{n+2}}\\ &\le c\left\|Q\left[f(V_{d,\eta}+\phi)-f(V_{d,\eta})-f'(V_{d,\eta})\phi\right]\right\|_{\frac{2n}{n+2}}\\ &\quad + c\left\|Qf(V_{d,\eta})-Q(\gamma_0U_{\delta,\xi})^p\right\|_{\frac{2n}{n+2}}+c\gamma_0^p\left\|\left[Q-Q(\xi_0)\right]U^p_{\delta,\xi}\right\|_{\frac{2n}{n+2}}.\end{aligned}$$

Using the mean value theorem, Lemma 2.3 and the Hölder inequality we have that, for some $t\in(0,1)$,

$$\begin{aligned}\left\|Q\left[f(V_{d,\eta}+\phi)-f(V_{d,\eta})-f'(V_{d,\eta})\phi\right]\right\|_{\frac{2n}{n+2}} &\le c\left\|\left[f'(V_{d,\eta}+t\phi)-f'(V_{d,\eta})\right]\phi\right\|_{\frac{2n}{n+2}}\\ &\le c\left\|f'(V_{d,\eta}+t\phi)-f'(V_{d,\eta})\right\|_{n/2}\|\phi\|_{2^*}\\ &\le c(\|\phi\|_{2^*}+\|\phi\|_{2^*}^{\frac{4}{n-2}})\,\|\phi\|_{2^*}\\ &\le c(\|\phi\|^2_{2^*}+\|\phi\|^p_{2^*}).\end{aligned}$$

Moreover, using Lemma 2.1 one can show that

$$\begin{aligned}&\left\|Qf(V_{d,\eta})-Q(\gamma_0U_{\delta,\xi})^p\right\|_{\frac{2n}{n+2}}\le c\left\|(P_\epsilon U_{\delta,\xi})^p-U^p_{\delta,\xi}\right\|_{\frac{2n}{n+2}}\\ &\le\left(c\int_{\Omega_\epsilon}\left|U^{p-1}_{\delta,\xi}(P_\epsilon U_{\delta,\xi}-U_{\delta,\xi})\right|^{\frac{2n}{n+2}}+c\int_{\Omega_\epsilon}\left|P_\epsilon U_{\delta,\xi}-U_{\delta,\xi}\right|^{p+1}\right)^{\frac{n+2}{2n}}\\ &\le c\delta,\end{aligned}\tag{2.14}$$

see inequality (6.4) in [9]. Finally, setting $y=\frac{x-\xi}{\delta}=\frac{x-\xi_0}{\delta}-\eta$ and $\widetilde{\Omega}_\epsilon:=\{y\in\mathbb{R}^n:\delta y+\xi\in\Omega_\epsilon\}$, and using the mean value theorem, for some $t\in(0,1)$ we obtain

$$\begin{aligned}\left\|\left[Q-Q(\xi_0)\right]U^p_{\delta,\xi}\right\|_{\frac{2n}{n+2}} &=\left(\int_{\widetilde{\Omega}_\epsilon}\left|Q(\delta y+\delta\eta+\xi_0)-Q(\xi_0)\right|^{\frac{2n}{n+2}}U^{p+1}(y)dy\right)^{\frac{n+2}{2n}}\\ &=\delta\left(\int_{\widetilde{\Omega}_\epsilon}\left|\langle\nabla Q(t\delta y+t\delta\eta+\xi_0),y+\eta\rangle\right|^{\frac{2n}{n+2}}U^{p+1}(y)dy\right)^{\frac{n+2}{2n}}\\ &\le c\delta.\end{aligned}\tag{2.15}$$

This proves statement (2.13).

Next we show that we may choose $\epsilon_0>0$ such that, for each $\epsilon\in(0,\epsilon_0)$, the operator

$$T^{\epsilon}_{d,\eta}:\{\phi\in K^{\epsilon,\perp}_{d,\eta}:\|\phi\|\le c\epsilon^{\frac{n-2}{n-1}}\}\to\{\phi\in K^{\epsilon,\perp}_{d,\eta}:\|\phi\|\le c\epsilon^{\frac{n-2}{n-1}}\}$$

is a contraction and, therefore, has a unique fixed point, as claimed.

If $\phi_1,\phi_2\in\{\phi\in K^{\epsilon,\perp}_{d,\eta}:\|\phi\|\le c\epsilon^{\frac{n-2}{n-1}}\}$, using again the mean value theorem we obtain

$$\begin{aligned}\left\|T^{\epsilon}_{d,\eta}(\phi_1)-T^{\epsilon}_{d,\eta}(\phi_2)\right\| &\le c\left\|f(V_{d,\eta}+\phi_1)-f(V_{d,\eta}+\phi_2)-f'(V_{d,\eta})(\phi_1-\phi_2)\right\|_{\frac{2n}{n+2}}\\ &=c\left\|\left[f'(V_{d,\eta}+(1-t)\phi_1+\phi_2)-f'(V_{d,\eta})\right](\phi_1-\phi_2)\right\|_{\frac{2n}{n+2}}\\ &\le c\left\|f'(V_{d,\eta}+(1-t)\phi_1+\phi_2)-f'(V_{d,\eta})\right\|_{\frac{n}{2}}\|\phi_1-\phi_2\|_{2^*}\end{aligned}$$

for some $t \in [0,1]$, and arguing as before we conclude that

$$\begin{aligned}\|f'(V_{d,\eta}+(1-t)\phi_1+\phi_2)-f'(V_{d,\eta})\|_{\frac{n}{2}} &\leq c\left(\|(1-t)\phi_1+\phi_2\|_{2^*}+\|(1-t)\phi_1+\phi_2\|_{2^*}^{\frac{4}{n-2}}\right)\\ &\leq c\left(\|\phi_1\|_{2^*}+\|\phi_2\|_{2^*}+\|\phi_1\|_{2^*}^{\frac{4}{n-2}}+\|\phi_2\|_{2^*}^{\frac{4}{n-2}}\right)\end{aligned}$$

Hence, if ϵ is sufficiently small, it follows that

$$\left\|T^{\epsilon}_{d,\eta}(\phi_1)-T^{\epsilon}_{d,\eta}(\phi_2)\right\| \leq \kappa\,\|\phi_1-\phi_2\|$$

with $\kappa \in (0,1)$.

Finally, a standard argument shows that $(d,\eta) \mapsto \phi^{\epsilon}_{d,\eta}$ is a C^1-map. This concludes the proof. □

Consider the functional $J_\epsilon : H^1_0(\Omega_\epsilon) \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{1}{2}\int_{\Omega_\epsilon}|\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} Q|u|^{p+1}.$$

It is well known that the critical points of J_ϵ are the solutions of problem (2.7). We define the reduced energy functional $\widetilde{J}^{\Gamma}_{\epsilon} : \Lambda^{\Gamma} \to \mathbb{R}$ by

$$\widetilde{J}^{\Gamma}_{\epsilon}(d,\eta) := J_\epsilon(V_{d,\eta}+\phi^{\epsilon}_{d,\eta}). \tag{2.16}$$

If $\Gamma = \{1\}$ is the trivial group, we simply write $\widetilde{J}_\epsilon$ instead of $\widetilde{J}^{\Gamma}_{\epsilon}$ and Λ instead of Λ^{Γ}.

Next we show that the critical points of $\widetilde{J}^{\Gamma}_{\epsilon}$ are Γ-invariant solutions of problem (2.7).

Proposition 2.5. *If $(d,\eta) \in \Lambda^{\Gamma}$ is a critical point of the function $\widetilde{J}^{\Gamma}_{\epsilon}$, then $V_{d,\eta} + \phi^{\epsilon}_{d,\eta} \in H^1_0(\Omega_\epsilon)^{\Gamma}$ is a critical point of the functional J_ϵ and, therefore, a Γ-invariant solution of problem* (2.7).

Proof. Assume first that Γ is the trivial group. Then $\Lambda = (0,\infty) \times \mathbb{R}^n$ and the statement is proved using similar arguments to those given to prove Lemma 6.1 in [6] or Proposition 2.2 in [9].

If Γ is an arbitrary closed subgroup of $O(n)$, then Λ^{Γ} is the set of Γ-fixed points in Λ of the action of Γ on the space $\mathbb{R} \times \mathbb{R}^n$ which is given by $g(t,x) := (t, gx)$ for $g \in \Gamma$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. By the principle of symmetric criticality [17, 22], if $(d,\eta) \in \Lambda^{\Gamma}$ is a critical point of the function $\widetilde{J}^{\Gamma}_{\epsilon}$, then (d,η) is a critical point of $\widetilde{J}_\epsilon : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, and the result follows from the previous case. □

3. The asymptotic expansion of the reduced energy functional

In order to find a critical point of $\widetilde{J}^{\Gamma}_{\epsilon}$ we will use the following asymptotic expansion of the functional $\widetilde{J}_\epsilon : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$.

Proposition 3.1. *The asymptotic expansion*

$$\widetilde{J}_\epsilon(d,\eta) = c_0 + Q(\xi_0)^{-\frac{2}{p-1}} F(d,\eta)\epsilon^{\frac{n-2}{n-1}} + o(\epsilon^{\frac{n-2}{n-1}})$$

holds true C^1-uniformly on compact subsets of Λ, where the function $F:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ is given by

$$F(d,\eta):=\begin{cases}\alpha d+\beta\frac{1}{(1+|\eta|^2)d}-\gamma\left\langle\frac{\nabla Q(\xi_0)}{Q(\xi_0)},\eta\right\rangle d & \text{if } n=3,\\ \beta\left(\frac{1}{(1+|\eta|^2)d}\right)^{n-2}-\gamma\left\langle\frac{\nabla Q(\xi_0)}{Q(\xi_0)},\eta\right\rangle d & \text{if } n\geq 4.\end{cases}\tag{3.1}$$

for some positive constants c_0,α,β and γ.

Proof. We write

$$\begin{aligned}J_\epsilon(V_{d,\eta}+\phi^\epsilon_{d,\eta})&=\frac{1}{2}\left\|V_{d,\eta}+\phi^\epsilon_{d,\eta}\right\|^2-\frac{1}{p+1}\int_{\Omega_\epsilon}Q\left|V_{d,\eta}+\phi^\epsilon_{d,\eta}\right|^{p+1}\\&=J_\epsilon(V_{d,\eta})+\gamma_0\int_{\Omega_\epsilon}(U^p_{\delta,\xi}-(P_\epsilon U_{\delta,\xi})^p)\phi^\epsilon_{d,\eta}\\&-\gamma_0^p\int_{\Omega_\epsilon}[Q-Q(\xi_0)]\,(P_\epsilon U_{\delta,\xi})^p\,\phi^\epsilon_{d,\eta}+\frac{1}{2}\left\|\phi^\epsilon_{d,\eta}\right\|^2\\&-\frac{1}{p+1}\int_{\Omega_\epsilon}Q\left(\left|V_{d,\eta}+\phi^\epsilon_{d,\eta}\right|^{p+1}-|V_{d,\eta}|^{p+1}-(p+1)V^p_{d,\eta}\phi^\epsilon_{d,\eta}\right).\end{aligned}$$

Then, using Hölder's inequality and inequalities (2.12), (2.14) and (2.15) we obtain

$$\begin{aligned}J_\epsilon(V_{d,\eta}+\phi^\epsilon_{d,\eta})&=J_\epsilon(V_{d,\eta})+O\left(\epsilon^{\frac{2(n-2)}{n-1}}\right)\\&=\gamma_0^2\left[\frac{1}{2}\int_{\Omega_\epsilon}U^p_{\delta,\xi}\,(P_\epsilon U_{\delta,\xi})-\frac{1}{p+1}\int_{\Omega_\epsilon}|P_\epsilon U_{\delta,\xi}|^{p+1}\right]\\&-\frac{1}{p+1}\gamma_0^{p+1}\int_{\Omega_\epsilon}[Q-Q(\xi_0)]\,|P_\epsilon U_{\delta,\xi}|^{p+1}+O\left(\epsilon^{\frac{2(n-2)}{n-1}}\right).\end{aligned}\tag{3.2}$$

Next, we compute the first summand on the right-hand side of equality (3.2). From Lemma 2.1 we have that

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\epsilon}U^p_{\delta,\xi}\,(P_\epsilon U_{\delta,\xi})-\frac{1}{p+1}\int_{\Omega_\epsilon}|P_\epsilon U_{\delta,\xi}|^{p+1}\\&=\frac{p-1}{2(p+1)}\int_{\Omega_\epsilon}U^{p+1}_{\delta,\xi}-\frac{1}{2}\int_{\Omega_\epsilon}U^p_{\delta,\xi}\,(P_\epsilon U_{\delta,\xi}-U_{\delta,\xi})-\frac{1}{p+1}\int_{\Omega_\epsilon}\left||P_\epsilon U_{\delta,\xi}|^{p+1}-U^{p+1}_{\delta,\xi}\right|\\&=\frac{p-1}{2(p+1)}\int_{\Omega_\epsilon}U^{p+1}_{1,0}-\frac{1}{2}\int_{\Omega_\epsilon}U^p_{\delta,\xi}\,(P_\epsilon U_{\delta,\xi}-U_{\delta,\xi})+o(\epsilon^{\frac{n-2}{n-1}})\\&=\frac{p-1}{2(p+1)}\int_{\mathbb{R}^n}U^{p+1}_{1,0}+\frac{1}{2}\int_{\mathbb{R}^n}U^p_{\delta,\xi}\Upsilon^\epsilon_{\delta,\xi}+o(\epsilon^{\frac{n-2}{n-1}}),\end{aligned}$$

where

$$\Upsilon^\epsilon_{\delta,\xi}(x):=\alpha_n\delta^{\frac{n-2}{2}}H(x,\xi)+\alpha_n\frac{1}{\delta^{\frac{n-2}{2}}(1+|\eta|^2)^{\frac{n-2}{2}}}\frac{\epsilon^{n-2}}{|x-\xi_0|^{n-2}}.\tag{3.3}$$

Setting $x = \xi + \delta y$ we have

$$\alpha_n \int_{\mathbb{R}^n} U^p_{\delta,\xi} \Upsilon^\epsilon_{\delta,\xi}$$

$$= \alpha_n \int_{\mathbb{R}^n} U^p_{\delta,\xi}(x)(\delta^{\frac{n-2}{2}} H(x,\xi))dx + \alpha_n \int_{\mathbb{R}^n} U^p_{\delta,\xi}(x) \left(\frac{1}{\delta^{\frac{n-2}{2}}(1+|\eta|^2)^{\frac{n-2}{2}}} \frac{\epsilon^{n-2}}{|x-\xi_0|^{n-2}} \right) dx$$

$$= \alpha_n \delta^{n-2} \int_{\mathbb{R}^n} U^p_{1,0}(y) H(\delta y + \delta\eta + \xi_0, \delta\eta + \xi_0) dy$$

$$+ \alpha_n \frac{1}{(1+|\eta|^2)^{\frac{n-2}{2}}} \int_{\mathbb{R}^n} U^p_{1,0}(y) \left(\frac{\epsilon^{n-2}}{\delta^{n-2}|y-\eta|^{n-2}} \right) dy$$

$$= \alpha_n \left(\int_{\mathbb{R}^n} U^p_{1,0} \right) H(\xi_0,\xi_0)\delta^{n-2}(1+o(1)) + \alpha_n g(\eta) \frac{1}{\delta^{n-2}} \epsilon^{n-2}(1+o(1)),$$

where the function $g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$g(\eta) := \frac{1}{(1+|\eta|^2)^{\frac{n-2}{2}}} \int_{\mathbb{R}^n} \frac{1}{|y-\eta|^{n-2}} U^p_{1,0}(y) dy.$$

Since $-\Delta U = U^p$ in $\mathbb{R}^n$, an easy computation shows that

$$g(\eta) = \frac{1}{(1+|\eta|^2)^{\frac{n-2}{2}}} U_{1,0}(\eta) = \alpha_n \frac{1}{(1+|\eta|^2)^{n-2}}.$$

To compute the second summand on the right-hand side of equality (3.2) we use the Taylor expansion

$$Q(\delta y + \xi_0 + \delta\eta) = Q(\xi_0) + \delta\langle \nabla Q(\xi_0), y + \eta\rangle + O(\delta^2(1+|y|^2))$$

to obtain

$$\int_{\Omega_\epsilon} [Q - Q(\xi_0)] \, |P_\epsilon U_{\delta,\xi}|^{p+1} = \int_{\Omega_\epsilon} [Q - Q(\xi_0)] \, U^{p+1}_{\delta,\xi} + o(\epsilon^{\frac{n-2}{n-1}})$$

$$= \int_{\widetilde{\Omega}_\epsilon} (Q(\delta y + \xi_0 + \delta\eta) - Q(\xi_0)) U^{p+1}_{1,0}(y) dy + o(\epsilon^{\frac{n-2}{n-1}})$$

$$= \delta \int_{\mathbb{R}^n} \langle \nabla Q(\xi_0), \eta\rangle U^{p+1}_{1,0}(y) dy + \delta \int_{\mathbb{R}^n} \frac{\langle \nabla Q(\xi_0), y\rangle}{(1+|y|^2)^n} dy + O\left(\epsilon^{\frac{2(n-2)}{n-1}}\right)$$

$$= \delta\langle \nabla Q(\xi_0), \eta\rangle \left(\int_{\mathbb{R}^n} U^{p+1}_{1,0} \right)(1+o(1)),$$

because $\int_{\mathbb{R}^n} \frac{\langle \nabla Q(\xi_0), y\rangle}{(1+|y|^2)^n} dy = 0$. Collecting all the previous information we obtain

$$\widetilde{J}_\epsilon(d,\eta) = J_\epsilon(V_{d,\eta} + \phi^\epsilon_{d,\eta})$$

$$= \begin{cases} c_0 + \gamma_0^2 \left(c_1 H(\xi_0,\xi_0) d + c_2 g(\eta) \frac{1}{d} - c_3 \left\langle \frac{\nabla Q(\xi_0)}{Q(\xi_0)}, \eta \right\rangle d \right) \sqrt{\epsilon} + o(\sqrt{\epsilon}) & \text{if } n = 3, \\ c_0 + \gamma_0^2 \left(c_2 g(\eta) \frac{1}{d^{n-2}} - c_3 \left\langle \frac{\nabla Q(\xi_0)}{Q(\xi_0)}, \eta \right\rangle d \right) \epsilon^{\frac{n-2}{n-1}} + o(\epsilon^{\frac{n-2}{n-1}}) & \text{if } n \geq 4, \end{cases}$$

as claimed. □

Proof of theorem 1.1. We will show that the function F defined in (3.1) has a critical point $(d_0, \eta_0) \in \Lambda^\Gamma = (0,\infty) \times (\mathbb{R}^n)^\Gamma$ which is stable under C^1-perturbations. Then, we deduce from Proposition 3.1 that the functional $\widetilde{J}^\Gamma_\epsilon$ has a critical point in Λ^Γ for ϵ small enough, so the result follows from Proposition 2.5.

Let $n=3$. Set $\zeta_0 := \frac{\nabla Q(\xi_0)}{Q(\xi_0)}$ and consider the half space $\mathcal{H} := \{\eta \in \mathbb{R}^3 : \alpha - \gamma\langle\zeta_0,\eta\rangle > 0\}$. For each $\eta\in\mathcal{H}$ there exists a unique $d=d(\eta)$, given by

$$d(\eta) = \sqrt{\frac{\beta}{(1+|\eta|^2)(\alpha-\gamma\langle\zeta_0,\eta\rangle)}} \in (0,\infty),$$

such that $F_d(d,\eta)=0$. Moreover, $F_{dd}(d(\eta),\eta)>0$ for any $\eta\in\mathcal{H}$. Consider the function $\widetilde{F}:\mathcal{H}\to\mathbb{R}$ defined by

$$\widetilde{F}(\eta) := F(d(\eta),\eta) = 2\beta^2\sqrt{\frac{\alpha-\gamma\langle\zeta_0,\eta\rangle}{1+|\eta|^2}}.$$

The point

$$\eta_0 := \left(\frac{\alpha-\sqrt{\alpha^2+\gamma^2\left|\zeta_0\right|^2}}{\gamma\left|\zeta_0\right|^2}\right)\zeta_0$$

is a strict maximum point of $\widetilde{F}$. Setting $d_0 := d(\eta_0)$ we deduce from Lemma 5.7 in [14] that (d_0,η_0) is a C^1-stable critical point of the function F. Note that, since $\xi_0\in\Omega^\Gamma$ and Q is Γ-invariant, $\nabla Q(\xi_0)\in(\mathbb{R}^n)^\Gamma$. Hence, $(d_0,\eta_0)\in\Lambda^\Gamma$.

If $n\geq 4$ arguing as in the previous case we easily conclude that, if

$$\eta_0 := -\frac{\nabla Q(\xi_0)}{|\nabla Q(\xi_0)|}, \qquad d_0 := \left(\frac{(n-2)\beta}{2^{n-2}\gamma}\frac{Q(\xi_0)}{|\nabla Q(\xi_0)|}\right)^{\frac{1}{n-1}},$$

then (d_0,η_0) is a C^1-stable critical point of the function F and $(d_0,\eta_0)\in\Lambda^\Gamma$. This concludes the proof. □

4. Final remarks

One may wonder whether Proposition 1.4 is also true in other dimensions. We show that this is not so.

If $N=k_1+k_2$ we write the elements of $\mathbb{R}^N$ as (y_1,y_2) with $y_i\in\mathbb{R}^{k_i}$, and the elements of $\mathbb{R}^{m+1}$ as (t,ζ) with $t\in\mathbb{R}$, $\zeta\in\mathbb{R}^m$.

Proposition 4.1. *Let $N=k_1+k_2$, $\mathcal{D}$ be an $[O(k_1)\times O(k_2)]$-invariant bounded smooth domain in $\mathbb{R}^N$ such that $0\notin\overline{\mathcal{D}}$, and $f\in C^0(\mathbb{R})$. Set*

$$\mathcal{U} := \{(t,\zeta)\in\mathbb{R}\times\mathbb{R}^m : \mathfrak{h}_{\mathbb{R}}(|y_1|,|y_2|) = (t,|\zeta|) \text{ for some } (y_1,y_2)\in\mathcal{D}\}$$

and let $u\in C^2(\mathcal{U})$, $u(t,\zeta)=\mathfrak{u}(t,|\zeta|)$, be an $O(m)$-invariant solution of equation

$$-\Delta u = \frac{1}{2|x|}f(u) \tag{4.1}$$

in $\mathcal{U}$. Then $v(y_1,y_2) := \mathfrak{u}(\mathfrak{h}_{\mathbb{R}}(|y_1|,|y_2|))$ is an $[O(k_1)\times O(k_2)]$-invariant solution of equation

$$-\Delta v = f(v) \tag{4.2}$$

in $\mathcal{D}$ if and only if $k_1=k_2=m$.

Proof. A straighforward computation shows that a function $v(y_1,y_2)=\mathfrak{v}(|y_1|,|y_2|)$ solves equation (4.2) in $\{(y_1,y_2)\in\mathcal{D} : y_1\neq 0,\ y_2\neq 0\}$ if and only if $\mathfrak{v}$ solves

$$-\Delta\mathfrak{v} - \frac{k_1-1}{z_1}\frac{\partial\mathfrak{v}}{\partial z_1} - \frac{k_2-1}{z_2}\frac{\partial\mathfrak{v}}{\partial z_2} = f(\mathfrak{v}) \tag{4.3}$$

in $\mathcal{D}_0 := \{z = (z_1, z_2) \in \mathbb{R}^2 : z_1, z_2 > 0,\ z_1 = |y_1|\,,\ z_2 = |y_2|,\ (y_1, y_2) \in \mathcal{D}\}$.

Similarly, a function $u(t,\zeta) = \mathfrak{u}(t, |\zeta|)$ solves equation (4.1) in $\{(t,\zeta) \in \mathcal{U} : \zeta \neq 0\}$ if and only if $\mathfrak{u}$ solves

$$-\Delta\mathfrak{u} - \frac{m-1}{x_2}\frac{\partial\mathfrak{u}}{\partial x_2} = \frac{1}{2\,|x|}f(\mathfrak{u}) \tag{4.4}$$

in $\mathcal{U}_0 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0,\ x_2 = |\zeta|\,,\ (x_1, \zeta) \in \mathcal{U}\}$.

Assuming that $\mathfrak{u}$ solves equation (4.4) in $\mathcal{U}_0$, we show next that $\mathfrak{v} := \mathfrak{u} \circ \mathfrak{h}_{\mathbb{R}}$ solves equation (4.3) in $\mathcal{D}_0$ if and only if $k_1 = k_2 = m$. A straightforward computation yields

$$\begin{aligned} &-\Delta\mathfrak{v} - \frac{k_1 - 1}{z_1}\frac{\partial\mathfrak{v}}{\partial z_1} - \frac{k_2 - 1}{z_2}\frac{\partial\mathfrak{v}}{\partial z_2} \\ &= 2\,|z|^2\left(-\Delta\mathfrak{u} - \left[\frac{k_1-1}{|z|^2} - \frac{k_2-1}{|z|^2}\right]\frac{\partial\mathfrak{u}}{\partial x_1} - \left[\frac{k_1-1}{|z|^2}\frac{z_2}{z_1} + \frac{k_2-1}{|z|^2}\frac{z_1}{z_2}\right]\frac{\partial\mathfrak{u}}{\partial x_2}\right). \end{aligned}$$

Note that $|z|^2 = |\mathfrak{h}_{\mathbb{R}}(z)|$. So if $\mathfrak{u}$ solves equation (4.4), we have that

$$-\Delta\mathfrak{v} - \frac{k_1 - 1}{z_1}\frac{\partial\mathfrak{v}}{\partial z_1} - \frac{k_2 - 1}{z_2}\frac{\partial\mathfrak{v}}{\partial z_2} = |\mathfrak{v}|^{p-2}\mathfrak{v}$$

if and only if

$$\frac{k_1-1}{|z|^2} = \frac{k_2-1}{|z|^2} \qquad \text{and} \qquad \frac{k_1-1}{|z|^2}\frac{z_2}{z_1} + \frac{k_2-1}{|z|^2}\frac{z_1}{z_2} = \frac{m-1}{z_1 z_2}$$

if and only if $k_1 = k_2 = m$. □

The argument given in [16] to prove Proposition 1.4 uses polar coordinates. Note that if we write $z_1 = r\cos\theta$, $z_2 = r\sin\theta$, $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$, then the Hopf map $x = \frac{1}{2}\mathfrak{h}_{\mathbb{R}}(z)$ becomes

$$\rho = \frac{1}{2}r^2, \qquad \varphi = 2\theta,$$

which is the map considered in [16].

References

[1] N. Ackermann, M. Clapp, A. Pistoia, Boundary clustered layers for some supercritical problems. *J. Differential Equations*, to appear.

[2] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain. *Comm. Pure Appl. Math.* **41** (1988), 253-294.

[3] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds.* London Mathematical Society Monographs. New Series, 29. The Clarendon Press, Oxford University Press, Oxford, 2003.

[4] M. Clapp, J. Faya, and A. Pistoia, Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents. *Calc. Var. Partial Differential Equations*, DOI: 10.1007/s00526-012-0564-6.

[5] M. del Pino, P. Felmer, M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries. *Bull. London Math. Soc.* **35** (2003), 513–521.

[6] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem. *Calc. Var. Partial Differential Equations* **16** (2003), 113–145.

[7] M. del Pino, M. Musso, F. Pacard, Bubbling along boundary geodesics near the second critical exponent. *J. Eur. Math. Soc. (JEMS)* **12** (2010), 1553–1605.

[8] J. Eells, A. Ratto, *Harmonic maps and minimal immersions with symmetries. Methods of ordinary differential equations applied to elliptic variational problems.* Annals of Mathematics Studies, 130. Princeton University Press, Princeton, NJ, 1993.

[9] Y. Ge, M. Musso, A. Pistoia, Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains. *Comm. Partial Differential Equations* **35** (2010), 1419-1457.
[10] J. Kazdan, F. Warner, Remarks on some quasilinear elliptic equations. *Comm. Pure Appl. Math.* **38** (1975), 557-569.
[11] S. Kim, A. Pistoia, Clustered boundary layer sign changing solutions for a supercritical problem. *J. London Math. Soc.*, to appear.
[12] S. Kim, A. Pistoia, Boundary towers of layers for some supercritical problems. Preprint arXiv:1302.1217.
[13] Y.Y. Li, On a singularly perturbed equation with Neumann boundary condition. *Comm. Partial Differential Equations* **23** (1998), 487–545.
[14] R. Molle, A. Pistoia, Concentration phenomena in elliptic problems with critical and supercritical growth. *Adv. Differential Equations* **8** (2003), 547–570.
[15] F. Pacella, A. Pistoia, Bubble concentration on spheres for supercritical elliptic problems. Preprint.
[16] F. Pacella, P.N. Srikanth, A reduction method for semilinear elliptic equations and solutions concentrating on spheres. Preprint. arXiv:1210.0782
[17] R.S. Palais, The principle of symmetric criticality. *Comm. Math. Phys.* **69** (1979), 19–30.
[18] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains. *J. Funct. Anal.* **114** (1993), 97–105.
[19] D. Passaseo, New nonexistence results for elliptic equations with supercritical nonlinearity. *Differential Integral Equations* **8** (1995), 577–586.
[20] S.I. Pohozhaev, Eigenfunctions of the equation $\Delta u+\lambda f(u)=0$. *Soviet Math. Dokl.* **6** (1965), 1408-1411.
[21] J. Wei, S. Yan, Infinitely many positive solutions for an elliptic problem with critical or supercritical growth. *J. Math. Pures Appl.* **96** (2011), 307–333.
[22] M. Willem, *Minimax theorems.* Progress in Nonlinear Differential Equations and their Applications **24**, Birkhäuser, Boston, 1996.
[23] J.C. Wood, *Harmonic morphisms between Riemannian manifolds.* Modern trends in geometry and topology, 397–414, Cluj Univ. Press, Cluj-Napoca, 2006.

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico
E-mail address: monica.clapp@im.unam.mx

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico
E-mail address: jorgefaya@gmail.com

Dipartimento di Metodi e Modelli Matematici, Universitá di Roma "La Sapienza", via Antonio Scarpa 16, 00161 Roma, Italy
E-mail address: pistoia@dmmm.uniroma1.it